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Reconstruction of domino tiling from its two orthogonal projections

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Abstract

We are interested in the reconstruction of a domino tiling of a rectangle from its two orthogonal projections. We give polynomial algorithms for some subproblems when all the dominoes are of the same type and prove NP-completeness results when there are three types of dominoes. When two types of dominoes are allowed, we give a polynomial-time transformation from a well-known open problem. © 2001 Elsevier Science B.V. All rights reserved.

Keywords: Domino tiling; Reconstruction problem; Polynomial-time algorithm; NP-completeness

1. Introduction

The reconstruction of a discrete picture from its projection is of primary importance in many fields of computer science such as pattern recognition, image processing, data compression, computer-aided tomography and it can also help for reconstructing pictures taken by an electron microscope which measures the number of each type of atom lying on each line in some direction [8, 2]. Many authors study the reconstruction of discrete pictures from its two orthogonal projections, the vertical and horizontal projections. Ryser [11] studied the reconstruction of a binary matrix, many authors study the reconstruction of polyominoes subject of particular constraints such as convexity [3, 6, 7], in [9, 5] authors study the reconstruction of a coloured matrix (each cell of the matrix is assigned with a colour).

We are interested in the reconstruction of the tiling of a rectangle with dominoes from its two orthogonal projections. The paper is organized as follows: in Section 2, we formally define our problem and give notations and previous results used in the

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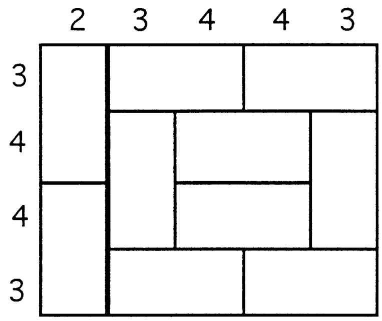

Fig. 1. A domino tiling and its orthogonal projections.

remainder of the paper. In Section 3, we give some properties of special subproblems namely horizontally and/or vertically decreasing tiling. In Sections 4 and 5, we design polynomial-time algorithms for these subproblems. In Sections 6 and 7, we are interested in coloured dominoes: we prove that the reconstruction of a tiling with two kinds of dominoes is harder than the reconstruction of a tricoloured matrix; for the case of three types of dominoes, we prove that the reconstruction problem is NP-complete even in the case of horizontally and vertically decreasing tiling. In the concluding section, we give some problems for which the complexity is still open.

2. Definitions, notations and previous results

A **domino** is a convex polyomino with two cells. We consider an $m \times n$ array, that will be tiling by dominoes, i.e. every cell of the array is covered and dominoes do not overlap. For each line $i, 1 \leqslant i \leqslant m$, respectively, column $j, 1 \leqslant j \leqslant n$, l_i denotes the number of dominoes covering at least one cell of line i, c_j denotes the number of dominoes covering at least one cell of column j, respectively. We say that l_i is the horizontal projection of line i and c_j is the vertical projection of column j. Fig. 1depicts a tiling of a 4×5 array and its two projections. A domino is **vertical** if it covers two cells (i,j) and $(i+1,j)$ on a same column; a domino is **horizontal** if it covers two cells (i,j) and $(i,j+1)$ on the same line.

We are interested in the following reconstruction problem: given two vectors of integers $\mathscr{L} = (l_1, \ldots, l_m)$ and $\mathscr{C} = (c_1, \ldots, c_n)$, is there a dominoes tiling of the array $m \times n$ with horizontal and vertical projections $l_i, 1 \leqslant i \leqslant m$, and $c_j, 1 \leqslant j \leqslant n$, respectively.

Remark. In order to avoid trivial instances, we may assume that $n/2 < l_i \leqslant n, 1 \leqslant i \leqslant m$ (if $l_i = n/2$ the line i is entirely filled with $n/2$ horizontal dominoes), and symmetrically, that $m/2 < c_i \leqslant m, 1 \leqslant i \leqslant n$. We make this assumption in the remainder of this paper.

We give below two definitions that allow us to define some particular subcases of the above reconstruction problem:

A tiling is **vertically decreasing** if and only if the vertical projections vector $\mathscr{C}$ satisfies $c_j \geqslant c_{j+1}, 1 \leqslant j \leqslant n-1$.

A tiling is **horizontally decreasing** if the horizontal projections vector $\mathscr{L}$ satisfies $l_i \geqslant l_{i+1}, 1 \leqslant i \leqslant m-1$.

We recall now some well-known results about the $(0,1)$-matrix reconstruction problem. The $(0,1)$-matrix reconstruction is defined as follows: $H=(h_1,\ldots,h_m)$ an m-vector of integers and $V=(v_1,\ldots,v_n)$ an n-vector of integers. The problem is to build an $m\times n$ $(0,1)$-matrix M such that $\sum_{j=1}^{n} M_{ij}=h_i, 1\leqslant i\leqslant m$, and $\sum_{i=1}^{m} M_{ij}=v_j, 1\leqslant j\leqslant n$.

Ryser [11, 4] gives necessary and sufficient conditions on H and V for such a matrix M to exist. We recall here this characterisation:

Let $r_1\geqslant r_2\geqslant\cdots\geqslant r_n$ be a sequence of positive integers. The Ferrer sequence derived from the r_i's is the sequence $r_1^*\geqslant r_2^*\geqslant\cdots\geqslant r_m^*$ defined as follows: $m=r_1$ and $r_k^*=\max_{1\leqslant i\leqslant n}\{i|r_i\geqslant k\}, 1\leqslant k\leqslant m$.

Ryser's Theorem. *Let $r_1\geqslant r_2\geqslant\cdots\geqslant r_p$ and $s_1\geqslant s_2\geqslant\cdots\geqslant s_q$ be two sequences of positive integers with $p\leqslant q$. There exists a $(0,1)$-matrix with horizontal projection $(r_1,\ldots,r_p)$ and vertical projection $(s_1,\ldots,s_q)$ if and only if: $\Sigma_{i=1}^{k} r_i^*\geqslant\Sigma_{j=1}^{k} s_j$, $1\leqslant k\leqslant q-1$, and $\Sigma_{i=1}^{p} r_i=\Sigma_{i=1}^{q} s_i$.*

The reconstruction of such an M can be obtained in polynomial time using standard flows algorithms for bipartite graphs, see [1] for such algorithms (some specific $O(nm)$ algorithm can also be designed).

3. Domino tiling properties

In this section we give some properties on the two projections of a domino tiling.

We denote the number of vertical dominoes on line i by v_i and the number of horizontal dominoes on line i by h_i.

Property 1. *Let l_i be the projection of line i. If a tiling exists then $h_i=n-l_i$ and $v_i=2l_i-n$.*

Proof. If a tiling exists then n cells of line i are covered. A horizontal domino covers two cells and a vertical domino covers one cell. Thus we have $2h_i+v_i=n$ and $h_i+v_i=l_i$, so the property holds. $\square$

We consider a vertical domino covering a cell of line i. So, two cases can occur: either the second cell of the domino lies on line $i+1$ or it lies on line $i-1$. In the first case, we say that the **domino begins** on line i and ends on line $i+1$. In the second case, the **domino ends** on line i, so it begins on $i-1$. Let us denote by w_i the number of dominoes beginning on line i.

Property 2. *If a tiling exists then $w_i=v_i-w_{i-1}\geqslant 0$, for $2\leqslant i\leqslant m$, and $w_1=v_1$.*

Proof. The property follows immediately from the definition of w_i's. □

We give now a property that occurs in a horizontally decreasing tiling:

Property 3. *In a horizontally decreasing tiling*:
(a) $h_i \leqslant h_{i+1}$ *and* $v_i \geqslant v_{i+1}, 1 \leqslant i \leqslant m-1$,
(b) $w_{2i} = 0, 1 \leqslant i \leqslant \frac{m}{2}$ *and* m *is even.*

Proof. (a) This is an immediate consequence of Property 1 and the definition of a horizontally decreasing tiling.

(b) We establish the first part of this property for w_2: following (a) and Property 2 we have $w_1 = v_1 \geqslant v_2 = w_2 + w_1$. Now, we can iterate in the same way for each $w_{2i}, 2 \leqslant i \leqslant m/2$. Following our assumption, $v_i > 0, 1 \leqslant i \leqslant m$ thus m is even. □

In the same manner, we define v_i as the number of vertical dominoes on column i, and η_i the number of horizontal dominoes on column i. We say that a horizontal domino begins on column i, if its second cell lies on column $i+1$. A domino ends on column i, if its second cell lies on $i-1$. We denote by ω_i the number of horizontal dominoes beginning on column i.

So we obtain the two properties given below:

Property 4. *Let* c_i *be the projection of column* i. *If a tiling exists then* $v_i = m - c_i$ *and* $\eta_i = 2c_i - m$. $\omega_i = \eta_i - \omega_{i-1} \geqslant 0$, *for* $2 \leqslant i \leqslant n$, *and* $\omega_1 = \eta_1$.

Property 5. *In a vertically decreasing tiling*:
(a) $\eta_i \leqslant \eta_{i+1}$ *and* $v_i \geqslant v_{i+1}, 1 \leqslant i \leqslant n-1$,
(b) $\omega_{2i} = 0, 1 \leqslant i \leqslant n/2$ *and* n *is even.*

4. Vertically and horizontally decreasing tiling

In this section, we give both, a characterisation and a reconstruction algorithm for a vertically and horizontally decreasing tiling.

The following property will allow us to give a characterisation of vertically and horizontally decreasing vectors $\mathscr{L}$ and $\mathscr{C}$ such that a domino tiling exists.

Property 6. *In a vertically and horizontally decreasing tiling, the block of four cells* $(i,j), (i+1,j), (i,j+1), (i+1,j+1)$, *with* i *and* j *odd, are covered with either two vertical dominoes beginning on cells* (i,j) *and* $(i,j+1)$, *or with two horizontal dominoes beginning on cells* (i,j) *and* $(i+1,j)$.

Proof. This property is a direct consequence of Properties 3 and 5 together. □

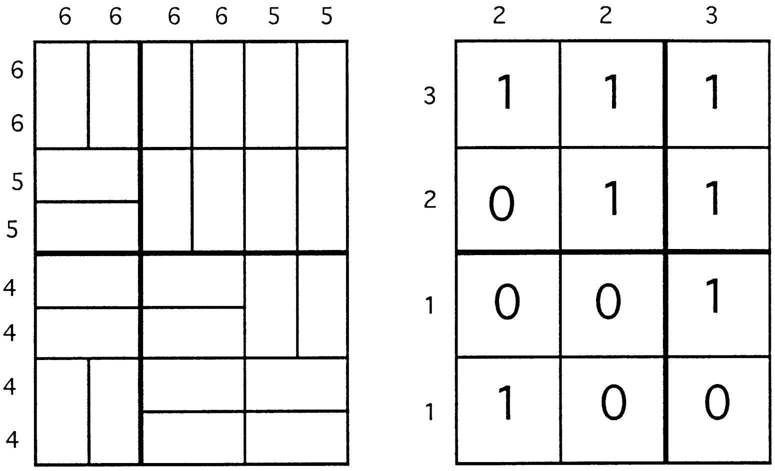

Fig. 2. A vertically and horizontally decreasing tiling and its associated $(0,1)$-matrix.

Using the above property, we can make an equivalence between the instances of the vertically and horizontally decreasing domino tilings and the instances of the $(0,1)$-matrix reconstruction problems. Each block of four cells $(i,j),(i+1,j),(i,j+1),(i+1,j+1)$, with i and j odd, is associated with coordinate $M_{\lceil i/2\rceil,\lceil j/2\rceil}$ of an $m/2\times n/2$ $(0,1)$-matrix M. With such a block covered by two vertical dominoes, is associated the value 1 for $M_{\lceil i/2\rceil,\lceil j/2\rceil}$, and analogously, with a block covered by horizontal dominoes is associated the value 0, for $M_{\lceil i/2\rceil,\lceil j/2\rceil}$. Taking $h_{\lceil i/2\rceil}=v_i/2$ for the number of 1 on line $\lceil i/2\rceil$ of M, and $v_{\lceil j/2\rceil}=v_j$ for the number of 1 on column $\lceil j/2\rceil$ of M, the instances of vertically and horizontally decreasing tiling reconstruction problem are on one-to-one correspondance with instances of $(0,1)$-matrix reconstruction problem. Fig. 2 shows this correspondance.

We can now establish the following two results:

Theorem 1. *For two vectors $\mathscr{L}$ and $\mathscr{C}$, the existence of a vertically and horizontally decreasing domino tiling can be done in linear time.*

The reconstruction problem can be solved with the same time complexity compared to the best algorithm for the maximum flow in bipartite network.

Proof. This is an immediate consequence of the equivalence with the $(0,1)$-matrix reconstruction since the correspondance between these two problems can be done with a linear time algorithm. $\square$

5. Horizontally decreasing or vertically decreasing tiling

We give in this section a polynomial algorithm that reconstruct a horizontally decreasing dominoes tiling if such a tiling exits. Trivially, this algorithm can be adapted for a vertically decreasing tiling.

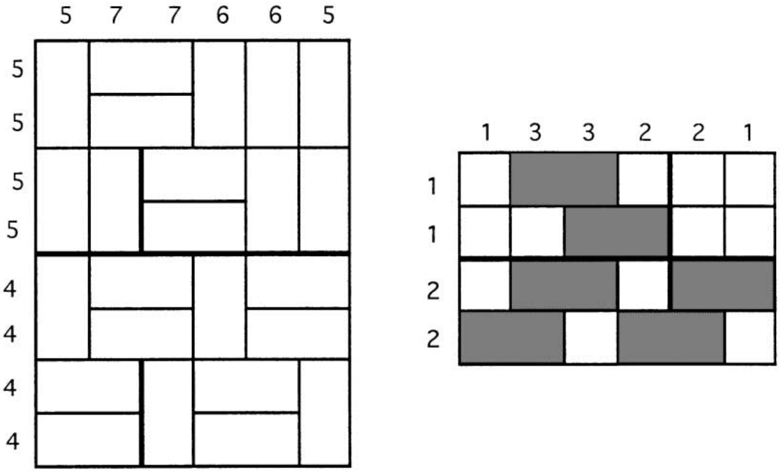

Fig. 3. A horizontally decreasing tiling and its associated horizontal dominoes packing.

Property 3 involves a particular structure for a horizontally decreasing tiling. Indeed, since vertical dominoes should begin on lines i with i odd, if a horizontal domino begins on cell (i,j) with i odd, there is a horizontal domino beginning on cell $(i+1,j)$ and vice versa. So every horizontal domino beginning on (i,j) with i odd, is matched with the horizontal domino beginning on $(i+1,j)$. The left-hand side of Fig. 3 illustrates this particular structure.

Using the particular structure of horizontally decreasing tilings, we will be interested in the horizontal dominoes packing reconstruction problem defined as follows: the dominoes are horizontal, for each line $i, 1 \leqslant i \leqslant n$, respectively, column $j, 1 \leqslant j \leqslant m$, α_i, the projection of line i, is the number of dominoes lying on line i, and β_j, the projection of column j, is the number of dominoes lying on column j. The problem is to reconstruct a packing, i.e. two dominoes without overlap, satisfying the two projections. The right-hand side of Fig. 3 shows such a packing. We will show that the solutions of this packing problem can easily be in one-to-one correspondance with the solutions of the horizontally decreasing tiling reconstruction problem. The transformation from the horizontally decreasing tiling reconstruction problem to the horizontal dominoes packing reconstruction problem is the following: the dimension of the array for the packing problem is $m/2 \times n$, $\alpha_i = h_{2i}, 1 \leqslant i \leqslant m/2$, and $\beta_j = v_j, 1 \leqslant j \leqslant n$. Fig. 3 illustrates this correspondance. Thus each pair of horizontal dominoes is associated with a horizontal domino of the packing problem. So the correspondance between the two problems is easy to establish and can be obtained in linear time.

We design now a polynomial-time algorithm that solves the horizontal dominoes packing reconstruction problem. We recall that ω_j denotes the number of horizontal dominoes beginning at column j (we suppose that trivial necessary conditions on the ω_j's are satisfied). The algorithm is greedy and builds a packing from left to right (from column $j = 1$ to $n-1$). At each step, we assign a line i for each domino beginning on column j. If a domino begins on a cell (i,j), we say that the cell $(i,j+1)$ is not allowed. α'_i is the number of dominoes not yet packed on line i, initially $\alpha'_i = \alpha_i$. The ω_j dominoes beginning on column j are assigned to the ω_j cells (i,j) such that the α'_i's are maximal among the allowed cells.

Lemma 1. *If a horizontal dominoes packing exists then the algorithm builds it.*

Proof. We suppose that a packing exists. We denote by $l(i)$ the column where the ith domino ends on line l for this packing (dominoes are counted from left to right). We will prove that there exists a packing with the same assignments as in the solution made by our algorithm. We use an induction on the column j.

We suppose that in the solution there is a domino beginning on cell $(k,1)$ with $\alpha'_k < \alpha'_l$ and that the cell $(l,1)$ is not covered, that is $k(1) < l(1)$. Firstly, we suppose that $k(1) < l(1), k(2) < l(2), \ldots, k(\alpha'_k) < l(\alpha'_k)$: swapping the assignments of lines k and l from column 1 to $l(\alpha'_k)$, we obtain another packing with a domino beginning on cell $(l,1)$. Fig. 4(a) illustrates this argument. Secondly, let r be the first index such that $l(r) \leqslant k(r)$: swapping the assignments of lines k and l from column 1 to $l(r) - 2$, we obtain another packing with a domino beginning on cell $(l,1)$ (see Fig. 4(b)). So, there exists a packing with the same set of cells $(i,1)$ covered as the set done by the algorithm.

By the induction hypothesis, we suppose that for every column $k, 1 \leqslant k \leqslant j-1$, if a domino begins at cell (i,k) in the packing built by the algorithm, then there is a packing such that there is a domino beginning on cell (i,k). We will prove that there is a packing such that the cells (i,j) where a domino begins are the same cells as those assigned with the algorithm. First, in any solution, if a domino begins on $(i,j-1)$ no domino can begin on (i,j); since such a cell (i,j) is not allowed, in the algorithm, no domino begins on (i,j). Suppose that in this packing there is a domino beginning on cell (k,j), with $\alpha'_k < \alpha'_l$, and that the cell (l,j) is allowed and not covered: we have $k(s+1) < l(t+1)$, where s and t are the numbers of dominoes beginning on columns $x < j$, on lines k and l, respectively, in the packing. So, swapping the assignments of lines k and l from column j as is done for the case $j = 1$, we obtain another packing with a domino beginning on cell (l,j). This completes the proof. □

Now, we are able to establish the following result.

Theorem 2. *Horizontally decreasing tilings or vertically decreasing tilings can be reconstructed in polynomial time.*

Proof. The transformation to the horizontal dominoes packing problem can easily be done in linear time. The algorithm for the horizontal domino packing reconstruction is polynomial. Thus, by virtue of Lemma 1, if a tiling exists, it can be reconstructed in polynomial time. □

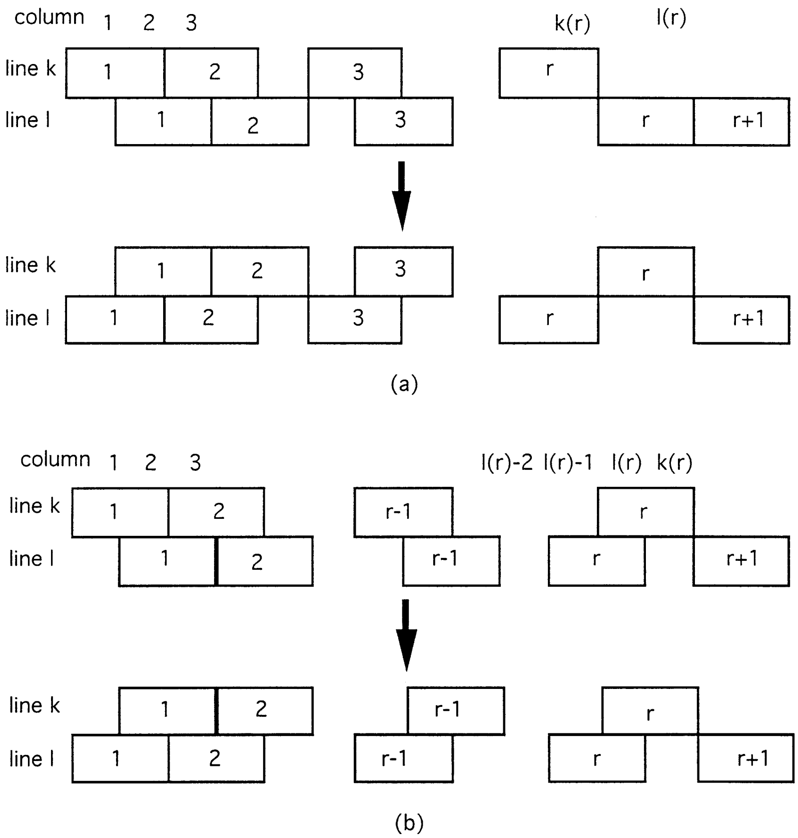

Fig. 4. Proof of Lemma 1.

6. Relation between the three colours and the bicoloured domino tiling reconstruction problems

In this section, we are interested in coloured dominoes. Each domino has its two cells white or its two cells black, so we call it a **white domino** or a **black domino**, respectively. For each horizontal, respectively, each vertical, projection, two nonnegative integers are given, the number of distinct dominoes and the number of distinct black dominoes, in the corresponding line, respectively, column. We establish that the problem of deciding if a bicoloured domino tiling exists is harder than the three colours reconstruction problem, even in the case of horizontally and vertically decreasing tiling.

The three colours reconstruction problem, 3-*colours* problem for short, is defined as follows:

Instance: Three nonnegative integers N-vectors: $H1=(H1_1,\dots,H1_N)$, $H2=(H2_1,\dots,H2_N)$, $H3=(H3_1,\dots,H3_N)$, and three nonnegative integers M-vectors: $V1=(V1_1,\dots,V1_M)$, $V2=(V2_1,\dots,V2_M)$, $V3=(V3_1,\dots,V3_M)$.

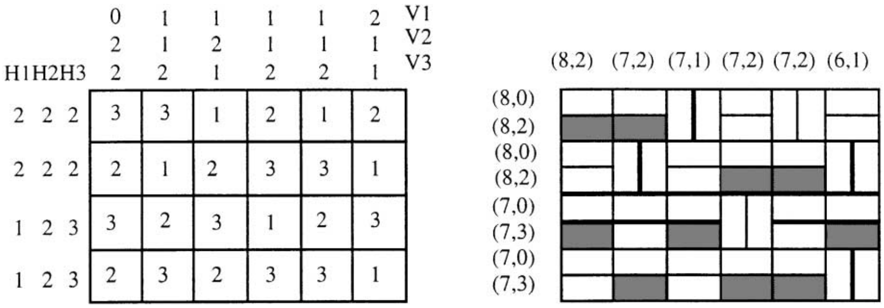

Fig. 5. The transformation from 3-colours to horizontally and vertically decreasing bicoloured domino tiling.

Question: Is there an $N \times M$ array having each of its cells coloured with colour C_1 or C_2 or C_3 such that: (a) for each line $i, 1 \leqslant i \leqslant N$, the number of cells coloured with C_1 is $H1_i$, the number of cells coloured with C_2 is $H2_i$ and the number of cells coloured with C_3 is $H3_i$; and (b) for each column $i, 1 \leqslant i \leqslant M$, the number of cells coloured with C_1 is $V1_i$, the number of cells coloured with C_2 is $V2_i$ and the number of cells coloured with C_3 is $V3_i$.

Theorem 3. 3-*colours* $\propto$ *horizontally and vertically decreasing bicoloured domino tiling.*

Proof. Fig. 5 illustrates the transfomation.

We can assume, without loss of generality, that the vector H_1 is such that $H1_1 \geqslant H1_2 \geqslant \cdots \geqslant H1_N$, and that the vector V_1 satisfies $V1_1 \leqslant V1_2 \leqslant \cdots \leqslant V1_M$.

The polynomial construction is as follows: the tiling has $2N$ lines and $2M$ columns; the number of dominoes on lines $2i-1$ and $2i$ is $2H1_i + H2_i + H3_i$, all the dominoes on lines $2i-1$ are white, and the number of black dominoes on lines $2i$ is $H3_i$; the number of dominoes on columns $2i-1$ and $2i$ is $V1_i + 2(V2_i + V3_i)$ and the number of black dominoes on columns $2i-1$ and $2i$ is $V3_i$.

Using the assumptions on H_1 and V_1, one can easily verify that we obtain a horizontally and vertically decreasing bicoloured domino tiling instance. So, combining Property 6 and the fact that only white dominoes stay on odd line, for every tiling, there are only three patterns that can cover the block $(2i-1, 2j-1), (2i-1, 2j), (2i, 2j-1), (2i, 2j)$: 2 vertical white dominoes associated with colour C_1, 2 horizontal white dominoes associated with colour C_2, and a horizontal black domino below a horizontal white domino associated with colour C_3.

Thus, equivalence between the instances of the two problems is easy to verify. $\square$

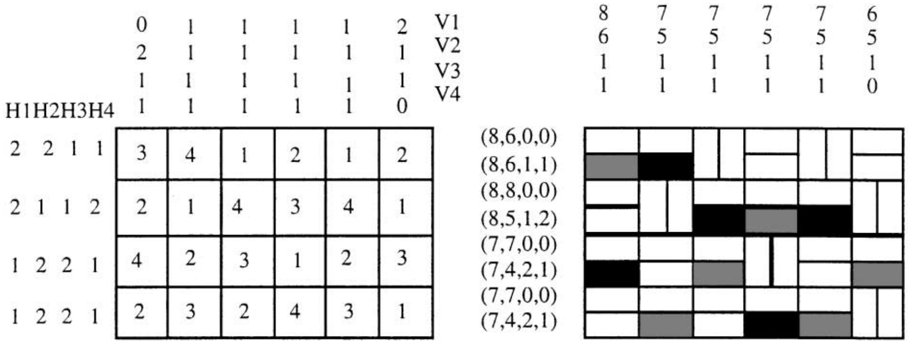

Fig. 6. The transformation from 4-colours to horizontally and vertically decreasing tricoloured domino tiling.

7. Tricoloured domino tiling reconstruction

Here, each domino has its two cells white or grey or black, so, as above we call it a **white domino** or a **grey domino** or a **black domino**. For each horizontal, respectively, each vertical, projection we have four nonnegative integers: the number of distinct dominoes, the number of distinct white dominoes, of grey dominoes, of black dominoes, respectively. Using a similar construction as for Theorem 3, we will prove the result stated below.

In order to prove this result, we give the definition of the four colours reconstruction problem, 4-*colours* problem for short:

Instance: four nonnegative integers N-vectors: $H1=(H1_1,\dots,H1_N)$, $H2=(H2_1,\dots,H2_N)$, $H3=(H3_1,\dots,H3_N)$, $H4=(H4_1,\dots,H4_N)$, and four nonnegative integers M-vectors: $V1=(V1_1,\dots,V1_M)$, $V2=(V2_1,\dots,V2_M)$, $V3=(V3_1,\dots,V3_M)$, $V4=(V4_1,\dots,V4_M)$.

Question: Is there an $N\times M$ array with every cell coloured with colour C_1 or C_2 or C_3 or C_4, such that for each line $i, 1\leqslant i\leqslant N$, the number of cells coloured with C_k is Hk_i, $k\in\{1,2,3,4\}$, and for each column $i, 1\leqslant i\leqslant M$, the number of cells coloured with C_k is Vk_i, $k\in\{1,2,3,4\}$.

This problem was proved to be NP-complete in [5].

Theorem 4. *The tricoloured domino tiling reconstruction problem is NP-complete even in the case of horizontally and vertically decreasing tiling.*

Proof. Using the same kind of construction as for the proof of Theorem 3, we show that 4-*colours* $\propto$ horizontal and vertical decreasing tricoloured domino tiling. Fig. 6 illustrates the transfomation.

We assume that the vector H_1 is such that $H1_1\geqslant H1_2\geqslant\cdots\geqslant H1_N$, and that the vector V_1 satisfies $V1_1\leqslant V1_2\leqslant\cdots\leqslant V1_M$.

The tiling has $2N$ lines and $2M$ columns; the number of dominoes on lines $2i-1$ and $2i$ is $2H1_i+H2_i+H3_i+H4_i$; all the dominoes on lines $2i-1$ are white; on lines

$2i$, the number of white dominoes is $2H1_i + H2_i$, the number of grey dominoes is $H3_i$ and the number of black dominoes is $H4_i$; the number of dominoes on columns $2i-1$ and $2i$ is $V1_i + 2(V2_i + V3_i + V4_i)$ and on columns $2i-1$ and $2i$, the number of white dominoes is $2N - (V3_i + V4_i)$, the number of grey dominoes is $V3_i$, and the number of black dominoes is $V4_i$.

Combining Property 6 and the fact that by our construction only white dominoes stay on odd line, for a tiling, there are only four patterns that can cover the block $(2i-1, 2j-1), (2i-1, 2j), (2i, 2j-1), (2i, 2j)$: either 2 vertical white dominoes associated with colour C_1, 2 horizontal white dominoes associated with colour C_2, a horizontal grey domino below a horizontal white domino associated with colour C_3, and a horizontal black domino below a horizontal white domino associated with colour C_4.

The equivalence between the instances of the two problems is easy to verify. □

8. Conclusion

In this paper, we have studied the reconstruction of domino tilings from their orthognonal projections. Under vertically and/or horizontally decreasing tiling assumption, we have designed polynomial-time algorithms. For the tricoloured tiling we have proved the NP-completeness of the problem even in the case of vertically and horizontally decreasing tiling. Yet some complexity problems are still unsolved: are the general reconstruction problems for monochromatic or bicoloured tilings polynomial or NP-complete?

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